Lectures on Challenging Mathematics

XC 7 Part 2

Functions, curves, matrices, and advanced 3-D geometry

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"Cogito ergo Sum" – "I think, therefore I am"

René Descartes (1596–1650)

"Success is not final, failure is not fatal, it is the courage to continue that counts."

Winston Churchill (1874–1965)

"I can see that without being excited, mathematics can look pointless and cold. The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani (1977–2017)

Contents

2.14 Ellipses (part 6)

1. Find an equation for each of the directrix of the following ellipses.

   (a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (b) $\dfrac{(x-2)^2}{16} + \dfrac{(y+5)^2}{25} = 1.$

2. Consider the ellipse $25x^2 + 16y^2 = 10000$. Line ℓ is tangent to the ellipse at a first quadrant point $P = (12, y_0)$. Find an equation of ℓ using a point near P.

3. (Continuation) Find an equation of ℓ using the reflection property of the tangent line to an ellipse.

4. (Continuation) Find an equation of the line that is tangent to the circle $x^2 + y^2 = 400$ at $R = (12, 16)$. Modify your line equation properly to obtain an equation of ℓ.

5. Consider the ellipse with foci $F_1(0,3)$ and $F_2(0,9)$ that passes through $P(2.4, 10)$. Find

   (a) its center and eccentricity;

   (b) a Cartesian equation of the ellipse;

   (c) a parametric equation of the ellipse;

   (d) an equation for a directrix of the ellipse;

   (e) the area of the region enclosed by the ellipse;

   (f) the maximum value of the area of the rectangle inscribed in the ellipse.

   (g) an equation of the line that is tangent to the ellipse at P. How many different methods you can use here?

2.15 Compositions of transformations and the products of matrices (part 2)

1. For each of the following coefficient matrices, describe the effect of its transformation:

   (a) $\begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}$

2. Write the coefficient matrix for a θ-degree clockwise rotation about the origin.

3. Compute the following product. Are you surprised by your answer?

$$\begin{bmatrix} \cos 57^\circ & -\sin 57^\circ \\ \sin 57^\circ & \cos 57^\circ \end{bmatrix} \begin{bmatrix} \cos 33^\circ & -\sin 33^\circ \\ \sin 33^\circ & \cos 33^\circ \end{bmatrix}$$

4. The result of reflecting across the line $y = -x$ and then rotating 330 degrees counterclockwise around the origin is an isometry $\mathcal{T}$.

   (a) Represent $\mathcal{T}$ by a 2×2 matrix. There is more than one way to find this unique matrix. Use the point $(1, 2)$ to check your answer.

   (b) How to describe $\mathcal{T}$ geometrically?

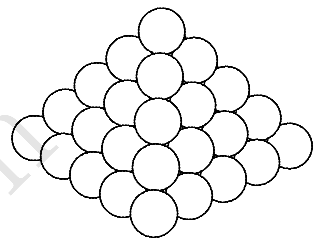
5. Write the coefficient matrix for the reflection across the line $y = \sqrt{3}x$.

3.14 Computations in 3-D geometry (part 2)

1. A cylinder of radius 4 and height h is inscribed in a sphere of radius 8. Find h.

2. Plane $9x - 2y - 6z = 8$ cuts the sphere $(x-1)^2 + (y-1)^2 + (z+2)^2 = 36$ to obtain cross section $\mathcal{R}$. What is the area of $\mathcal{R}$?

3. As shown in the diagram to the right, a grocer has 1015 spherical grapefruit, which are to be stacked in a square pyramid — one in the top layer, four in the next layer, etc. How many layers will the completed pyramid have? The diameter of each grapefruit is 6 inches. Find the height of the completed pyramid.



4. Write an expression for the volume of the spherical shell formed between two concentric spheres, the inner one of radius r, the outer one of radius R. Factor your answer so that it has the form $\frac{4}{3}\pi \cdot (\textit{trinomial}) \cdot (\textit{binomial})$. In this situation, what is the meaning of the *binomial*? What can be said about the value of the *trinomial* when the *binomial* has a very small value? Make a conjecture concerning the *surface area* of a sphere of radius R.

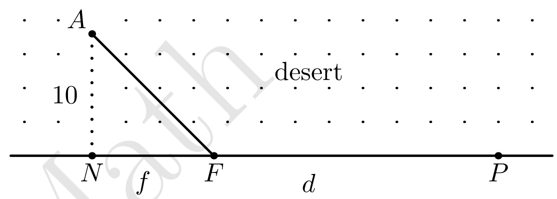
5. Both the slant height and the base diameter of a cone are 12 inches. How far is it from a point on the base circle to the diametrically opposite point on the circle, if it is required that the path must lie on the lateral surface of the cone?

   Query: Is the path with minimum distance a planar path; that is, it is entirely lying in one plane?

5.4 Snell's law (part 2)

1. Alex the geologist is located at point A in the desert, a km from a long, straight road. (Both sides of the road are desert.) On the road, Alex's jeep can do 50 kph, but in the desert sands, it can only manage 30 kph. Alex is very thirsty, and knows that there is a gas station, denoted by P, d km down the road (from the nearest point N on the road) that has ice-cold Pepsi. Let T denote the minimum time T_m for Alex to get to the gas station. Clearly, T depends on a and d so we can write $T = T(a, d)$.

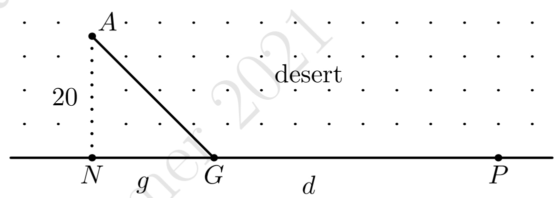
   In this part of the problem, we set $a = 10$. For every given value of d, to obtain $T(10, d)$, Alex has to go directly through desert from A to a point F on the road and then directly on the road from F to P. Set $NF = f$.



   (a) Sketch an accurate graph of f versus d (with horizontal axis representing d).

   (b) Sketch an accurate graph of $T(10, d)$ versus d (with horizontal axis representing d).

   In this part of the problem, we set $a = 20$. For every given value of d, to obtain $T(20, d)$, Alex has to go directly through desert from A to a point G on the road and then directly on the road from G to P. Set $NG = g$.



   (c) Sketch an accurate graph of g versus d (with horizontal axis representing d).

   (d) Sketch an accurate graph of $T(20, d)$ versus d (with horizontal axis representing d).

2. (Continuation)

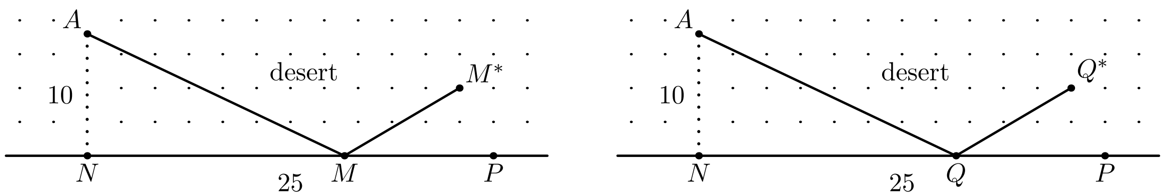
   (a) In what respects are the graphs of f and g versus d similar to each other? In what respects do they differ from each other?

   (b) In what respects are the graphs of $T(10, d)$ and $T(20, d)$ versus d similar to each other? In what respects do they differ from other?

   (c) How to explain your observations algebraically? (Applying a proper *substitution* such as $g = 2f$ in the expression of $T(20, d)$ could be very helpful.)

3. Alex the geologist is located at point A in the desert, 10 km from a long, straight road. (Both sides of the road are desert. See the left-hand side diagram shown below.) On the road, Alex's jeep can do 50 kph, but in the desert sands, it can only manage 30 kph. Alex is very thirsty, and knows that there is a gas station, denoted by P, 25 km down the road (from the nearest point N on the road) that has ice-cold Pepsi. Let M be the midpoint of segment NP. Find a point M^* in the desert such that it takes Alex an equal amount of time traveling from

A to M^* via M (staying completely in the desert) and from A to P via M. How many such points are there? Find the locus $\mathcal{C}_M$ of all such points.



4. (Continuation) Let Q be a variable point on segment NP. (See the right-hand side diagram shown above.) Find the locus $\mathcal{C}_Q$ of all the points Q^* satisfying the following property: It takes Alex an equal amount of time traveling from A to Q^* via Q (staying completely in the desert) and from A to P via Q.

5. (Continuation) In geometry, an *envelope* of a family of curves in the plane is a curve that is tangent to each member of the family at some point. You can check out the illustrations of this definition at `en.wikipedia.org/wiki/Envelope_(mathematics)`.

   Find the envelopes of the family of curves $\mathcal{C}_Q$ where Q moves along segment NP.

   Determine the fastest route for Alex to follow. One piece of the envelope is more helpful than the other.